KNOT COMPLEMENTS AND 3-MANIFOLDS

BY

LECHENG SU

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Applied Mathematics
in the Graduate College of the
University of Illinois Urbana-Champaign, 2021

Urbana, Illinois

Adviser:

Professor Igor Mineyev

# ABSTRACT

The study of 3-manifold groups has reached a big success in the last few decades, with the proof of geometrization theorem by Perelman, and several virtually compact special theorems by Wise, etc. In this paper, we will explore several aspects of 3-manifold groups and see what it shows for the knot complement as a compact 3-manifold. We will also apply some combinatorial method in three manifold topology to the study of the presentation of the knot complement.

*To Igor, a true geometric group theorist.*
*To my teammates Ali Guo and Josh Utley.*
*To my parents and friends, for their love and support.*

# TABLE OF CONTENTS

# LIST OF FIGURES

# CHAPTER 1

# INTRODUCTION

The goal of this paper is to keep track of some recent (the last half century) progress in the study of 3-manifold topology, more precisely, 3-manifold groups. One can view most of this paper as an introduction to 3-manifold topology , knot theory and some essential techniques for studying 3-manifolds. We also want to show what some important theorems can tell about a knot complement as a compact 3-manifold with toroidal boundary.

*Readers are expected to have a little experience with 3-manifold topology and geometric group theory.*

In chapter 2, we will give basic definitions for hyperbolic space, and generalize to hyperbolic manifolds and Gromov-hyperbolicity.

In chapter 3, we will discuss many basic notions and constructions about 3-manifolds, like essentially embedded surfaces, basic types of 3-manifolds. We will also introduce cube complex as a geometric group theory notion, which is a strong tool for studying 3-manifold groups, and is mainly developed by Wise, Haglund, Agol, Groves, Manning etc.

In chapter 4, we will develop basics in knot theory, and find some basic properties of the knot complement as a 3-manifold.

In chapter 5, we will keep our eyes on some insightful theorems, like geometrization, and virtually compact special. We will also explore what can be shown for knot complements, and what can be done in future study of knot complements.

Here are some pieces of important theorems in history.
First one is a theorem that is true for all tame knot complements:

**Theorem 1.1.** *The complement $C$ of a non-trivial knot is a Haken manifold and is determined by its fundamental group together with its peripheral system.*

The following theorem shows the relation between the fundamental group of hyperbolic 3-manifolds and cube complex.

**Proposition 1.1.** *Hyperbolic 3-manifold groups act properly on cubings*

Given a closed orientable 3-manifold $M^3$, the 0th and 3rd homology groups are both isomorphic to $\mathbb{Z}$, and the first is abelianized by $\pi_1(M^3)$. The fundamental group carries the most information for closed 3-manifolds. Furthermore, the next theorem shows that group-wise, closed 3-manifolds are more "basic", unless a 3-manifold group is not finitely presentable.

**Theorem 1.2.** *For $n \geq 4$, every finitely presentable group is isomorphic to the fundamental group of a closed hyperbolic $n$-manifold.*

The theorems given above will be thoroughly discussed in chapter 5 once we have defined everything we need for stating these theorems.

Note that it is most important for readers to understand the definitions and constructions and have a picture(one or two examples) in mind so that they can understand what the theorems show.

# CHAPTER 2

# HYPERBOLIC SPACES

As is known to all, we have several models for hyperbolic spaces, like a sphere with unit imaginary radius, the upper half space with hyperbolic Riemannian metric, or a unit ball with a conformal Riemannian metric. For sake of simplicity, we only introduce the first one, because other models are better for studying the local structures of hyperbolic spaces.

## 2.1 Basics in metric space

**Definition 2.1.** *An **isometry** between metric spaces* $(X, d)$ *and* $(X^{'}, d^{'})$ *is a bijection* $f : X \to X^{'}$ *such that* $d^{'}(f(x), f(y)) = d(x, y)$ *for any* $x, y \in X$. $X$ *and* $X^{'}$ *are said to be **isometric**.*

**Definition 2.2.** *Given a metric space* $(X, d), x, y \in X$, *a **geodesic** from* $x$ *to* $y$ *is a map* $g : [0, l] \subset \mathbb{R} \to X$ *such that* $g(0) = x, g(l) = y$, *and that* $d(g(t), g(t^{'})) = |t - t^{'}|$ *for any* $t, t^{'} \in [0, l]$.

Note that in this definition, the geodesic is parametrized by its arc length, i.e., $d(x, y) = l$. Geodesic rays and geodesic lines are defined similarly on the domains $[0, +\infty)$ and $\mathbb{R}$.

**Definition 2.3.** *A metric space* $(X, d)$ *is a **(uniquely) geodesic metric space** if any two points can be joined by a (unique) geodesic.*

**Definition 2.4.** *A subset* $C$ *of a metric space* $(X, d)$ *is **convex** if every pair of points inside* $C$ *can be joined by a geodesic in* $X$ *and the image of the geodesic is a subset of* $C$.

## 2.2 Hyperbolic spaces

This section and the next section mainly follow definitions in Ratcliffe's book[1], and secretly follow this book[2]. We will introduce the hyperboloid model for hyperbolic n-space, because we need the metric in this definition to define its isometry group later.

The (real) hyperbolic n-space relative to the Lorentzian (n+1)-space is similar to the n-sphere relative to the real (n+1)-space. They are defined mathematically the same way. The Lorentzian n-space is the real n-space with an unusual inner product-the Lorentzian inner product.

**Definition 2.5.** *Let $x, y$ in $\mathbb{R}^n$, the **Lorentzian inner product** of $x$ and $y$ is defined to be*

$$x \circ y = -x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

*The **Lorentzian norm** of $x$ in $\mathbb{R}^n$ is an imaginary number*

$$||x|| = (x \circ x)^{\frac{1}{2}}.$$

*If $||x|| = 0$, then $x$ is called **light-like**,*
*if $||x|| < 0$, then $x$ is called **time-like**,*
*if $||x|| > 0$, then $x$ is called **space-like**.*

The names for the three kinds of vectors above come from general relativity, where events in $\mathbb{R}^{1,3}$ can only happen in the light cone, which means they should be light-like vectors.

With this definition of inner product, we can define the special inner product space:

**Definition 2.6.** *The **Lorentzian n-space** $\mathbb{R}^{1,n-1}$, is the real n-space $\mathbb{R}^n$, together with the Lorentzian inner product.*

The hyperboloid model for real hyperbolic space is then defined as a half sphere with imaginary radius, as a 2-dimensional case in Figure 2.1:

**Definition 2.7.** *A sphere with imaginary radius is defined by*

$$F^n = \left\{x \in \mathbb{R}^{n+1} : ||x||^2 = -1\right\}$$

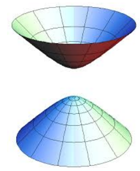

Figure 2.1: The hyperboloid model

*A **hyperbolic n-space** is the positive subset:*

$$\mathbb{H}^n = \{x \in F^n : x_1 > 0\}$$

*or as the sphere modulo antipodal map:*

$$\mathbb{H}^n = F^n / \{x \sim -x\}$$

## 2.3 Hyperbolic manifolds

A hyperbolic manifold can be defined as a manifold with hyperbolic structure, which means it is locally the hyperbolic space, and the maps in the charts are isometries.

**Definition 2.8.** *An $(X, G)$-**atlas** for an n-manifold $M$ is a family of functions $\Phi = \{\phi_i : U_i \to X\}_{i \in \mathcal{I}}$ satisfying:*

- *For each $i$, $U_i$ is an open connected subset of $M$.*
- *For each $i$, $\phi_i$ is a homeomorphism onto an open subset of $X$.*
- *$\{U_i\}_{i \in \mathcal{I}}$ covers $M$.*
- *For each $i, j \in \mathcal{I}$, if $U_i \cap U_j \neq \emptyset$, then the function*

  $$\phi_i \phi_j^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

  *agrees in a neighborhood of each point of its domain with an element of $G$.*

This definition above can be thought of as a natural inheritance of $C^k$-structure in the study of $C^k$-manifolds, with a slight more restriction that the transition map is an element in the designated group $G$.

Here is a similar theorem to the theorem in smooth manifold stating that if there is a smooth atlas, there is a maximal smooth atlas.

**Theorem 2.1.** *If there exists an $(X, G)$-atlas for a manifold $M$, then there exists a unique maximal $(X, G)$-atlas for $M$.*

We refer to reader this book[1] for proof of this theorem.

**Definition 2.9.** *The maximal atlas in Theorem 2.3.1 is called an* $(X, G)$***-structure***.

**Definition 2.10.** *An* $(X, G)$***-manifold*** *is an n-manifold $M$ together with an $(X, G)$-structure for $M$.*

A hyperbolic manifold is a special case of $(X, G)$-manifolds:

**Definition 2.11.** *A* ***hyperbolic n-manifold*** *is an n-manifold with a hyperbolic structure: $(\mathbb{H}^n, Isom(\mathbb{H}^n))$, where $Isom(\mathbb{H}^n)$ is the isometry group for hyperbolic n-space $\mathbb{H}^n$.*

There are several alternate definitions for hyperbolic $n$-manifolds once the hyperbolic $n$-space is defined. One of them is as a complete Riemannian $n$-manifold of constant sectional curvature -1. Every complete, 1-connected, $n$-manifold of constant sectional curvature -1 is isometric to $\mathbb{H}^n$. Therefore, a hyperbolic $n$-manifold can also be written as $\mathbb{H}^n/\Gamma$, i.e., the hyperbolic n-space modulo a torsion-free, discrete subgroup of $Isom(\mathbb{H}^n)$. The universal cover of a hyperbolic 3-manifold is isometric to $\mathbb{H}^3$[3], so to sum up, each hyperbolic 3-manifold $M$ can also be written as $\mathbb{H}^3/\pi_1(M)$

## 2.4 Gromov hyperbolicity and hyperbolic groups

We take a slight detour here to talk about a more generalized idea of hyperbolicity, Gromov-hyperbolicity.

There are several equivalent definitions of Gromov-hyperbolic metric space and its boundary, using triangles. Readers with further interest can refer to Mineyev's papers[4][5] for more detailed and technical discussion. But what needs notice is that those definitions require the space to be a geodesic metric space.

To introduce the notion of Gromov-hyperbolicity, we need a definition for Gromov product in metric spaces. Note that the definition for Gromov product does not require the space to be geodesic. The Gromov product can be thought of as a generalization of relative distance.

**Definition 2.12.** *For a metric space $(X, d)$, pick a base-point $p \in X$, and two points $x, y \in X$, we define the* ***Gromov product*** *of $x$ and $y$ to be:*

$$(x|y)_p = \frac{1}{2}(d(x,p) + d(y,p) - d(x,y))$$

This definition gives a "distance function" indeed because of triangle inequality of the metric of $X$. The Gromov product measures how far this triangle inequality is from equality[2].

**Definition 2.13.** *A metric space $(X, d)$ is* ***$\delta$-hyperbolic*** *in the sense of Gromov, if for all quadruples $x, y, z, p \in X$, we have*

$$(x|y)_p \leq min((x|z)_p, (y|z)_p) - \delta$$

**Example 2.1.** *The real line is 0-hyperbolic in the sense of Gromov.*

The following two lemmas show the equivalence of Gromov hyperbolicity and other definitions of hyperbolicity in geodesic metric spaces

**Lemma 2.1.** *If a metric space $(X, d)$ is $\delta$-hyperbolic in the sense of thin or slim triangles, then it is $3\delta$-hyperbolic in the sense of Gromov.*

The above lemma still does not require the metric space to be geodesic.

**Lemma 2.2.** *If a geodesic metric space is $\delta$-hyperbolic in the sense of Gromov, then it's $2\delta$-hyperbolic in the sense of thin/slim triangles.*

**Corollary 2.1.** *By lemma 2.4.1 and 2.4.2, for a geodesic metric space, Gromov-hyperbolicity is equivalent to other definitions using triangles.*

**Definition 2.14.** *A group $G$ is* ***word-hyperbolic*** *if its Cayley graph is a Gromov-hyperbolic metric space.*

# CHAPTER 3

# NOTIONS IN 3-MANIFOLDS AND CUBE COMPLEXES

The study of 3-manifold topology has reached a big success in the last few decades. The two important turning points are the Geometrization theorem by Perelman and the virtually compact special theorem by Agol. In this chapter, we will introduce some definitions before and after geometrization, along with some notions for understanding the virtually compact special theorem, like cube complex.

## 3.1 3-manifolds and embedded surfaces

The fundamental group carries the most information about the topology of a 3-manifold[6]. Consider a closed orientable 3-manifold, the first homology group equals the abelianization of the knot group, and the second homology group is equal to $\mathbb{Z}^{p_1}$, where $p_1$ is the first Betti number, the zeroth and third homology groups are equal to $\mathbb{Z}$(consider connected 3-manifolds). The cohomology groups can be determined using Poincaré Duality.

In order to write the whole story about 3-manifold topology, one probably need 200 pages, like this book[7]. So in this section, we will introduce notions and constructions that we find the most interesting and might potentially be related to the knot complement. Readers can read this section with the knot complement as an example in head.

We will partially refer to this book[7] for important definitions in 3-manifolds.

**Definition 3.1.** *An embedded submanifold $\Sigma \subset M$ in a manifold is* ***proper*** *if $\partial\Sigma = \Sigma \cap \partial M$, and the intersection is transverse.*

In this definition, we mix the submanifold with its image via embedding. Figure 3.1 is the correct picture to think of when considering a properly

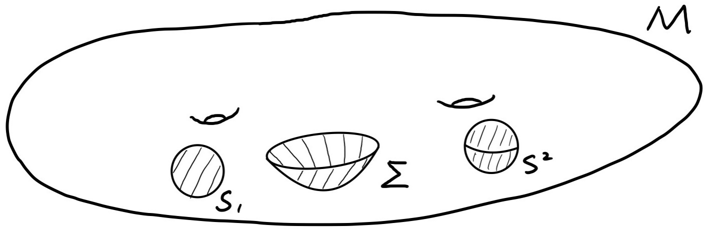


Figure 3.1: Types of embedded surfaces in a 3-manifold

embedded surface in a 3-manifold, where $\Sigma$ is a properly embedded surface in $M$, while $S_1$ is a compressible on the boundary, and $S^2$ is an embedded sphere that is also not proper.

Note that this definition works for compact manifolds, for general topological or smooth manifolds, we only require the map to be proper, that means, preimage of a compact subset is compact. We will further show in chapter 4 and 5 that the knot complement is a compact manifold, and its toral boundary is a properly embedded surface.

**Definition 3.2.** *An orientable 3-manifold $M$ is* ***prime*** *if it cannot be decomposed into non-trivial connected sum of two manifolds. $M$ is* ***irreducible*** *if every properly embedded 2-sphere in $M$ bounds a 3-ball in $M$.*

The definition means if $M \cong M_1 \# M_2$, then either $M_1$ or $M_2$ is $S^3$. Connected sum will be defined later in section 2, right now readers can think of it as cutting two small pieces from two manifolds and gluing them together. The definitions of prime and irreducible for orientable 3-manifolds are the same except for one case: $S^1 \times S^2$. One can think of this 3-manifold as a thickened $S^2$ with its two boundary components identified[8]. Therefore, one can think of orientable, irreducible 3-manifolds as prime 3-manifolds except for this one case.

**Definition 3.3.** *A topological space $X$ is* ***aspherical*** *if $X$ is connected and $\pi_i(X) = 0$ for all $i > 1$. A space $X$ is a $K(\pi, 1)$**-space** if $\pi \cong \pi_i(X)$, and $X$ is aspherical.*

$X$ has a contractible universal cover if it is connected and aspherical.

We then turn back our interest in surfaces in 3-manifolds and define what an essential surface is, following the book by Schulten[9]. To introduce what an essential surface in a 3-manifold is, we need to rule out three different situations: boundary parallel, compressible and boundary-compressible.

The next definition build on a general topological 3-manifold:

**Definition 3.4.** *A connected, properly-embedded, orientable surface in a 3-manifold (or a surface contained in the boundary of a 3-manifold) $\Sigma \subset M$ is* ***incompressible*** *if it is not homeomorphic to $S^2$ or $D^2$, and the homomorphism induced by inclusion $\pi_1(\Sigma) \to \pi_1(M)$ is injective.*

An incompressible surface with many path components is defined similarly at each component. This is the algebraic definition for an (orientable) incompressible surface using fundamental group. However, it can also be defined geometrically.

**Definition 3.5.** *Let $\Sigma$ be a connected, properly-embedded, orientable surface in a 3-manifold $M$ (or $\Sigma \subset \partial M$), and let $D$ be a properly embedded disk in $M$ such that $D \cap \Sigma = \partial D$. $D$ is said to be a* ***compression disk*** *for $\Sigma$ if $\partial D$ does NOT bound a disk on $\Sigma$. If $\Sigma$ admits a compression disk, then it is* ***compressible****. If $\Sigma$ is not $S^2$ or $D^2$ and it does NOT admit a compression disk, then it's* ***incompressible****.*

In this definition, a surface is incompressible if it cannot be compressed by a disk. The former definition implies the latter, and when we talk about fundamental group of 3-manifolds, the former definition is easier for calculation. However, the latter definition is more geometric and imaginable. Fortunately, for orientable surfaces, those two definitions coincides. Therefore, we can keep the second definition in mind, and use the first definition in practice.
Here's a piece of useful corollary for knot complement, which we will use in chapter 5:

**Corollary 3.1.** *The tubular neighborhood of a nontrivial knot is incompressible in the knot exterior.*

To define what boundary compressible is, we need to define what an essential arc is, which is a one-dimensional analog of an essential surface, and can be done in only one definition.

**Definition 3.6.** *A simple arc $\alpha$ in a surface $\Sigma$ is an* ***essential arc*** *if there is no arc $\beta$ in $\partial\Sigma$ such that $\alpha \cup \beta$ is a closed 1-manifold that bounds a disk in $\Sigma$.*

**Definition 3.7.** *Let $M$ be a 3-manifold, a surface $\Sigma \subset M$ is* ***boundary compressible*** *if there is an essential simple arc $\alpha \subset \Sigma$ and an essential simple arc $\beta \subset \partial M$, such that $\alpha \cup \beta$ is a closed 1-manifold that bounds a disk $D$ in $M$, and $int(D) \cap \Sigma$ is empty.*

The definition is almost the same form as the essential arc definition. If you cannot find such pair of arcs, then the surface is boundary incompressible.

We need the last piece of special case, which is boundary parallel:

**Definition 3.8.** *A surface $\Sigma$ in a connected 3-manifold $M$ is* ***boundary parallel****, if it is separating and a component of $M \setminus \Sigma$ is homeomorphic to $\Sigma \times I$, where $I \subset \mathbb{R}$ is the unit interval.*

This means, a surface is boundary parallel, if it can be isotoped to the boundary of $M$.

**Definition 3.9.** *A properly embedded surface $\Sigma$ in a 3-manifold $M$ is* ***essential*** *if it is incompressible, boundary incompressible and not boundary parallel.*

According to this definition, a 2-sphere in $M$ is essential if it does not bound a 3-ball.

Another interesting type of 3-manifolds are Haken manifolds, introduced by Haken in 1961[10]. Haken also proved in this paper that this type of 3-manifolds can be repeated cut along incompressible surfaces until the remaining pieces are 3-sphere.
Geometrically speaking, Haken manifolds are sufficiently large. Thurston[11] proved that Haken manifolds admit geometrization.

Before we introduce the notion of Haken, we take a step back and look at the definition of a peripheral system. The settings for the definition is complicated: Let $M$ be a 3-manifold and $S_1, \ldots, S_r$ its boundary components. Suppose the boundary components are incompressible, properly embedded surfaces, the embedding $i : S_i \to M$ induces a homomorphism from $S_i$ to $M$ if we take base-point $p \in S_i$: $i_* : \pi_1(S_i, p) \to \pi_1(M, p)$ .

**Definition 3.10.** *The* ***peripheral system*** *or peripheral structure of $M$ consists of $\pi_1(M)$ together with the conjugacy classes of $i_*(\pi_1(S_j, p))$, where $j = 1, 2, \ldots, r$.*

Each $\pi_1(S_j, p)$ in this definition is viewed as a subgroup of $\pi_1(M)$.

**Definition 3.11.** *An orientable 3-manifold is* ***Haken*** *if it is compact, irreducible, and has a properly embedded incompressible surface.*

The "sufficiently large" kind of corresponds to the last condition, embedded incompressible surface. Some people remove the irreducible condition when defining Haken(actually it is not present in the original definition[12]), that's because they are considering both orientable and unorientable cases. In the unorientable case, we may also change the irreducibility into $\mathbb{P}^2$-irreducibility.

We can also use embedded torus to study the topology of a 3-manifold, especially when the manifold contains an essential torus. Thurston[11] defined the notion atoroidal in two ways: geometrically and homotopically. We will use the definition similar to the homotopical one.

**Definition 3.12.** *A 3-manifold $M$ is* ***atoroidal*** *if any map $T \to M$ from a torus to $M$ that induces a monomorphism $\pi_1(T) \to \pi_1(M)$ can be homotoped into the boundary of $M$.*

Algebraically speaking, the fundamental group $\pi_1(M)$ should not contain $\mathbb{Z} \times \mathbb{Z}$ as a subgroup. Geometrically speaking, according to Thurston, every incompressible torus embedded in $M$ should be isotopic to a boundary component, or $M$ should not contain any essential torus. The study of embedded tori in 3-manifolds leads to a beautiful theorem called JSJ-decomposition theorem, which will be mentioned in chapter 5.

## 3.2 Constructing/gluing 3-manifolds

There are several ways to construct a new manifold from several old ones, one of them is by connected sum. A connected sum of two manifolds can be viewed as gluing together two contractible parts of two manifold to form a new manifold. Take two $n$-manifolds $M_1$ and $M_2$, and delete an $n$-ball from both manifolds:

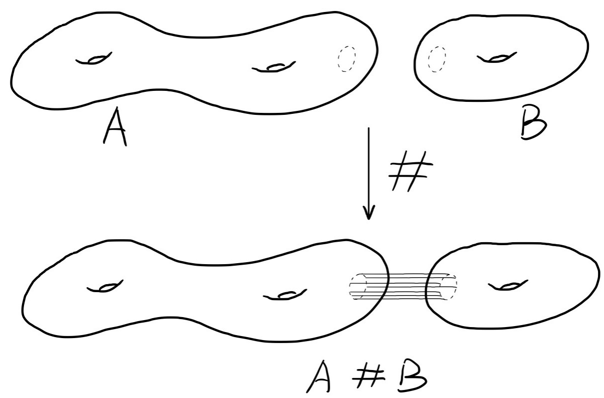


Figure 3.2: Connected sum of two genus g surfaces

**Definition 3.13.** *A **connected sum** of $M_1$ and $M_2$ is by gluing two manifolds along the boundaries of two balls. The result is denoted by $M_1 \# M_2$.*

Note that if both $M_1$ and $M_2$ are orientable, then we identify the spheres by an orientation reversing homeomorphism. In the case of two genus g surfaces, the connected sum is shown in Figure 3.2.

**Definition 3.14.** *A **handlebody** $H_g$ of genus $g$ is the homeomorphic image of the regular neighborhood of a bouquet of $g$ circles in $S^3$*

$$B_g = S^1 \vee \ldots \vee S^1 \tag{3.1}$$

Additionally, this bouquet is a strong deformation retract of this handlebody, and the retract induces an isomorphism between $\pi_1(H_g)$ and $\pi_1(B_g)$. It is well known that a piecewise linear 3-manifold can be triangulated. For any triangulation of an orientable closed 3-manifold $M^3$, the regular neighborhood of the 1-skeleton is a handlebody $H_g$ of genus $g$, with its complement $H_g^{'} = \overline{M^3 - H_g} \approx H_g$

**Definition 3.15.** *The pair $(H_g, H_g^{'})$ is called a **Heegaard splitting** of $M^3$.*

The above definition leads to the following theorem:

**Theorem 3.1.** *The fundamental group of a closed 3-manifold admits a balanced presentation. More precisely, if the manifold possesses a Heegaard splitting of genus $g$ then the fundamental group can be presented by $g$ generators and $g$ defining relators.*

This theorem shows that the fundamental group of a closed 3-manifold that admits such a splitting can be studied using genus g surfaces.

## 3.3 Cube complexes

Cube complex was originally a purely group theory construction, using pairs of groups, constructing a wallspace, and then cubulating the wallspace[13]. But later on, people in 3-manifold topology realized that cube complex, as a combinatorial thing, can be used to study 3-manifolds, if the fundamental group of a 3-manifold acts on a cube complex[7]. The biggest recent result using cube complex is the virtually compact special theorem, which is used to solve the virtual Haken conjecture and virtual fibering conjecture[14]. This result answers Thurston's question 18[11]. Some further discussion about virtual Haken conjecture will be in chapter 5

An enormous development of cube complex is due to several pieces of work by Wise and Haglund[15][16], but the original full construction is due to Sageev[13], who uses a wallspace structure on pairs of groups. For basic definitions in geometric group theory, we recommend Druţu's book[2] to readers , as the basics for some notions in cube complex.

Cube complexes can be viewed as a generalized construction of tree, or as a cell complex. Therefore, it is a combinatorial notion.

**Definition 3.16.** *An **n-cube** is a homeomorphic image of the product of unit intervals* $[-1,1]^n$. *A **face** in an n-cube is a restriction of some coordinate to* $\pm 1$. *A **cube complex** is a cell complex whose cells are all cubes and the attaching maps are defined on faces.*

Faces are also cube complexes. The homeomorphism in the definition can be given more restriction, depending on the need, like being an isometry. An equivalent definition for cube complex is by gluing together cubes along faces, which coincides with the definition of cell complex. A square complex is a 2-dimensional cube complex. A combinatorial map between cube complexes $X$ and $Y$ is a map such that the image of each $k$-cube in $X$ is a $k$-cube in $Y$. We also denote cubes in $0, 1, 2$ dimensions by vertices, edges and squares.

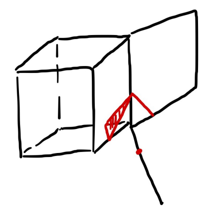

Figure 3.3: The link of a cube complex(red)

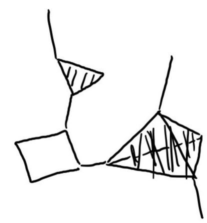

Figure 3.4: A flag complex

The link of a cube complex is a local simplicial complex around a vertex, which is kind of like taking a cube and cutting it along the corner and get a simplex, and the union of all these simplices form the link of a vertex. The link condition was originally introduced by Gromov in his famous 87 paper[17]. Since this part is about developing a geometric picture for understanding topology of 3-manifolds, we will only give a vague definition for link. Readers with further interest can refer to Bridson[18] for further discussion.

**Construction 3.1.** *Given a cube complex $X$ and choose a vertex $v$ in $X$, $v$ should be a "corner point" for one or several cubes. Choose a point near $v$ on each edge that touches $v$. Every $n$ points in the same $n$-cube span a $(n-1)$-simplex that cuts the cube into two parts. The union of all such simplices is the link at $v$, denoted by $Link(v)$.*

A link can be thought of as an "$\epsilon$-sphere" around a vertex in a cube complex.

**Definition 3.17.** *A **flag complex** is a simplicial complex such that whenever there are $n+1$ pairwise adjacent points(which means each pair is joined by an edge), they MUST span an $n$-simplex.*

This definition means, in a flag complex, there should be no "triangles", because whenever there's a triangle, it should span a 2-simplex, and higher dimensional cases are similar. Note that if a graph is a flag, then there should not be a cycle of $length < 4$.

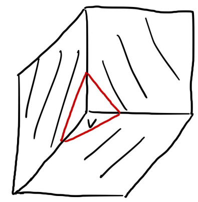


Figure 3.5: A positively curved square complex

**Definition 3.18.** *A cube complex $X$ is **non-positively curved**, or **NPC**, if $Link(v)$ is a flag complex for each $v \in X^0$. $X$ is $CAT(0)$ if it is non-positively curved and simply connected.*

The definition of $CAT(0)$ for more general metric spaces is by Euclidean comparison triangles, see Bridson[18]. It is also proven in this book that for a cube complex, those two definitions for $CAT(0)$ are equivalent. Each $CAT(0)$ cube complex is an universal cover of a non-positively curved cube complex. Figure 3.5 shows a positively curved cube complex, because the link around the vertex $v$ is a triangle, not a flag.

**Definition 3.19.** *A **mid-cube** of a cube is a subset of a cube by taking one of the intervals to be 0. A **hyperplane** of a cube complex $X$ is maximally spanned cube complex in which all cubes are mid-cubes.*

This definition means that the intersection of the hyperplane with each cube in $X$ is either *empty* or a *mid-cube* of that cube. Additionally, a hyperplane can also be represented by its dual oriented edges, which can be viewed as an equivalence class of oriented edges with "parallel" edges identified. See Farley[19] for more discussion.

With the definition of hyperplane in hand, we can finally define what a special cube complex is. Basically, the definition is just by ruling out several "bad" embeddings of hyperplanes.

**Definition 3.20.** *A **special cube complex** is a non-positively curved cube complex that does NOT have the following four kinds of hyperplanes:*

1. *An immersed hyperplane is not embedded.*
2. *A hyperplane is not 2-sided.*
3. *A hyperplane self-osculates.*

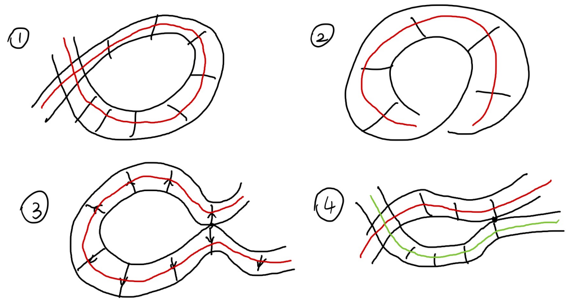


Figure 3.6: Four pathologies of hyperplanes that contradict specialness

*4. Two hyperplanes inter-osculates.*

Figure 3.6 corresponds to the four pathologies mentioned in the definition.

**Example 3.1.** *A **Dehn complex** is constructed in following steps:*

- *Given a knot $k \subset S^3$, project it onto some plane in $S^3$. The projection separates the plane into several regions. Paint the plane with checkerboard coloring, which means neighboring regions have opposite colors.*
- *Two vertices located above and below the plane.*
- *An edge for each region. All edges are connecting the two vertices. An edge go from up to down when the region is black, down to up when white.*
- *A two cell(square) is attached at each crossing along its neighboring regions.*

*The projection in this construction should be a regular projection, which we will see in chapter for a detailed definition. A Dehn complex of a knot projection is an NPC cube complex if and only if the projection is* prime *and* alternating*[20]. The Dehn complex of a regular trefoil projection is shown in Figure 3.7.*

Here's another important example introduced by Salvetti[21]:

**Example 3.2.** *A **Salvetti complex** $S_\Gamma$ of a finite graph $\Gamma = (V(\Gamma), E(\Gamma))$ is constructed in the following steps:*

- *$S_\Gamma$ has a single vertex $v_0$;*

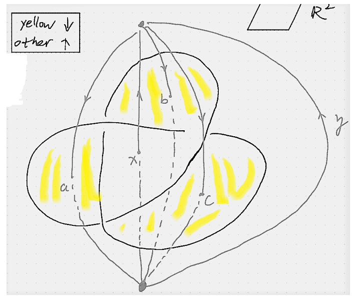


Figure 3.7: Dehn complex of a trefoil(1-skeleton)

- *$S_\Gamma$ has an edge $e_v$ for each element $v$ in $v \in V(\Gamma)$(edge is oriented, so $e_v$ and $\bar{e_v}$ are different edges);*
- *$S_\Gamma$ has a square attached along the edge $e_u e_v \bar{e_u} \bar{e_v}$ when $u$ and $v$ are joined by an edge in $E(\Gamma)$;*
- *$S_\Gamma$ has an $n$-cube whenever the $(n-1)$-skeleton is there.*

*It is easy to check that the Salvetti complex satisfies the link condition, so it is an NPC cube complex.*

We can also use Salvetti complex further for constructing a canonical local isometry for each cube complex, from the cube complex to the Salvetti complex of its hyperplane graph.

**Construction 3.2.** *A **hyperplane graph** of a cube complex $X$ is a graph with vertex set containing all hyperplanes of $X$, and the edge set corresponding to intersections of hyperplanes.*

**Proposition 3.1.** *Let $X$ be a special cube complex with finitely many hyperplanes. Then there is a locally isometric immersion from $X$ to the Salvetti complex of the hyperplane graph.*

A picture of a hyperplane graph to is shown in Figure 3.8. The fundamental group of a Salvetti complex is a famous type of group in geometric group theory, called right-angled Artin group(RAAG):

**Definition 3.21.** *Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph. A **RAAG** of this graph is defined as a group with generator set equals $V(\Gamma)$, and relators as commutators $v_i v_j \bar{v_i} \bar{v_j}$ or $[v_i, v_j]$ when $v_i$ and $v_j$ are joined by an edge in $E(\Gamma)$*

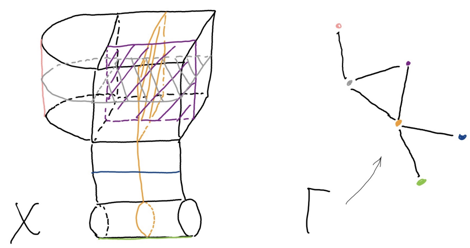


Figure 3.8: Hyperplane graph $\Gamma$ of a cube complex $X$

A RAAG can be viewed as a generalization of right-angled Coxeter group, which has a relation with braid group. Braid representation is another big topic in knot theory, but it will not be introduced in this article, readers with further interest about these relations can refer to this article[22].

The following proposition is clear:

**Proposition 3.2.** *The fundamental group of a Salvetti complex is isomorphic to a RAAG.*

With this proposition and Proposition 3.1, we can say that a special cube complex group, or a special group, embeds in a RAAG. Another way of saying this is a RAAG is "virtually" a special group. This is one of the key observations in the study of special cube complexes.

# CHAPTER 4

# DEFINITIONS AND NOTIONS IN KNOTS

Knot theory is a big subject that inspires many fields such as 3-manifold topology and geometric group theory. Thus, it is impossible to cover all basic notions in one chapter, so we will only introduce the aspects that we care the most. We will mostly treat knot and its complement in $S^3$ in a topological flavor.

## 4.1 The basics

In this section, we briefly introduce the basic definitions in knot theory.
This part will follow the book[23] by Burde et al, which means notions and constructions are introduced in a rather topological manner.

**Definition 4.1.** *A* ***knot*** *is an embedding of a circle $S^1$ into Euclidean 3-space $\mathbb{R}^3$, or the 3-sphere, $S^3$.*

Higher dimensional knots can be thought of as embeddings of $S^k$ into $S^{n+k}$. But throughout this paper, we will focus our discussions on $S^1 \subset S^3$. We always identify the knot with its image in $S^3$. Topologically, $S^3$ can be viewed as $\mathbb{R}^3$ with a point at infinity, i.e., the one point compactification of $\mathbb{R}^3$.

Two knots can be viewed as equivalent if there is some kind of homeomorphism from one to the other. However, it is hard to explicitly write such a homeomorphism. So we should find another way to show this equivalency. Homotopy between two knots seems a good canditate, yet not enough. Therefore, the idea of ambient isotopy is introduced, which means that the homotopical property of two knots should be carried along with its ambient space, $S^3$.

**Definition 4.2.** *Two knots $k_1$ and $k_2$ are* ***equivalent*** *if they are* ***ambient isotopic****, which means, there is a homotopy $H : S^3 \times I \to S^3$, such that each*

*$H(-,t) : S^3 \to S^3$ is a homeomorphism, and that*

$$H(k_1, 0) = k_1,\ H(k_1, 1) = k_2,$$

*where $I$ denotes the unit interval $[0, 1]$.*

**Definition 4.3.** *A knot is **trivial** if it is equivalent to $S^1$.*

**Definition 4.4.** *A knot is called **tame** if it is equivalent to a piecewise linear(p.l.) embedding in $S^3$, **wild** if otherwise.*

A tame knot can also be viewed as a simple closed polygonal curve in $S^3$[6]. From now on, we use the term knot instead of tame knot, which means we do not care about wild knots.

The connected sum of two knots is a bit different from the connected sum of two manifolds. The latter, defined in chapter 3, is constructed by deleting one ball in each manifold and gluing them together, while the former is by cutting along a trivial part of each knot and then concatenating them together. In terms of the knot complement, which we will define later, when doing connected sum of two knots, the complement of the resulting knot is generally not the connected sum of the original two knot complements.

**Definition 4.5.** *A knot is **composite** if it is a connected sum of two non-trivial knots. A knot is **prime** if it is not composite.*

The definition for a prime knot is similar to the definition for a prime 3-manifold, which means it cannot be decomposed into two simpler pieces of knots.

The most efficient way to represent a knot is to use the knot projection, which is by *projecting a knot $k$ onto a plane(projection plane) in $S^3$*. A point at the projection plane is called a *multiple point* if its preimage contains more than one element.

**Definition 4.6.** *A projection is called **regular** if there are only finitely many multiple points, each of which is a double point(with exactly two points in the preimage), and no vertex is mapped to a double point no matter which p.l. partition we choose for the knot. A **crossing** is a double point in the projection. The **crossing number** of a knot is the minimal number of crossings over all regular projections.*

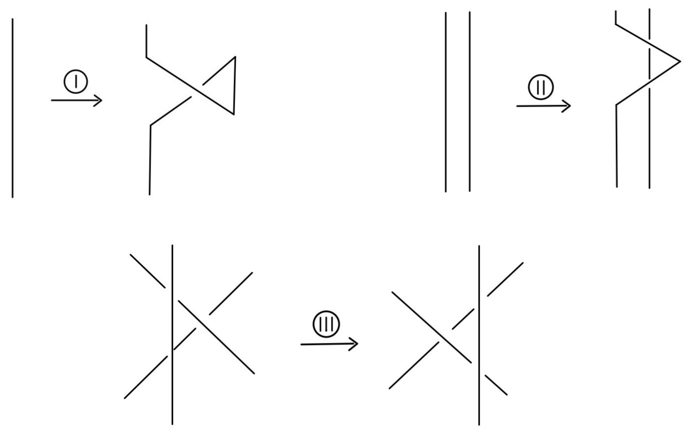


Figure 4.1: The Reidemeister moves

We can restrict our interest in regular knot projections because of the following theorem.

**Theorem 4.1.** *Every tame knot admits a regular projection/a polygonal projection.*

**Definition 4.7.** *Two knot projections/diagrams are* ***equivalent*** *if we can go from one to the other using a finite sequence of (inverse) Reidemeister moves, which is defined in Figure 4.1.*

**Definition 4.8.** *A* ***torus knot*** *is a knot that is an embedded submanifold* $S^1$ *on a torus.*

This is the easiest definition of a torus knot, building on the notion of embedded submanifold. It is of course true that this knot on the torus should have no crossings because it will contradict with the fact that it is an embedding.

Torus knot can be classified using Heegaard splitting and a pair of numbers $(a, b)$ of intersection with meridian and longitude on the splitting tori[23]. One can show that the crossing number of a torus knot $(a, b)$ is equal to $min\{a(b-1), b(a-1)\}$.

The following proposition rules out the possibility for a torus knot complement to be hyperbolic.

**Proposition 4.1.** *A torus knot complement does not admit a hyperbolic structure.*

Readers can refer to Burde's book[23] or Purcell's book[24] for a proof.

**Definition 4.9.** *Let $k$ be a knot that stays in an unknotted solid torus $T$. Let $e : T \to S^3$ be a homeomorphism onto its image such that $e(T)$, the image, is a regular neighborhood of a knot $c \subset S^3$. Then $s = e(k)$ is the* ***satellite knot*** *of $c$, where $c$ is called the* ***companion knot****, $(k, T)$ the* ***pattern*** *of $s$.*

The companion knot $c$ is simply as if you see the knot $s$ from far away, where you forget some local twist on the knot.

The definition of satellite knot also includes composite knot, i.e., a connected sum of two knots is a special case of a satellite knot from one orbiting another.

## 4.2 The Wirtinger presentation

The topology of the knot complement completely determines the topology of the knot[23]. Tietez[25] conjectured that two knot types are equal if and only if their complements are homeomorphic, which was proven[26] by Gordon and Luecke in 1989. To classify knot complements, we need to introduce the knot group - the fundamental group of the knot complement. The knot group encodes the most information of the knot complement because we can show later in this chapter that the knot complement is a compact 3-manifold.

In the very first, we shall give the definition of the knot complement. One thing to be clear is that the existence of the regular neighborhood of a (tame) knot, which is homeomorphic to a solid torus $S^1 \times D^2$. Studying the knot is the same thing as studying the knot complement in $S^3$. The actual "knot complement" is not convenient for study, so some topologically invariant change should be made.

**Definition 4.10.** *Given a knot $k \subset S^3$, the* ***complement of*** *$k$ is defined to be $S^3 \setminus V(k)$, where $V(k)$ is the regular neighborhood of $k$ in $S^3$.*

Regular neighborhood in this context means tubular neighborhood with no overlap. This can be done because the knot is tame. Compared to the *actual knot complement*, this definition does not change any topological property of it. Furthermore, the knot complement defined above is essentially a deformation retract of the *actual knot complement*.

**Definition 4.11.** *The* ***knot group*** *is defined to be the fundamental group of the knot complement.*

The fundamental group of a non-trivial tame knot is a finitely-generated group. The following theorem[27] shows that every knot group has a finite presentation:

**Theorem 4.2.** *If a group is generated by $n$ of its elements, then it is a quotient group of a free group of rank $n$.*

Additionally, because of this theorem, every finitely generated group has a presentation. Using projection technique together with van Kampen theorem, we can explicitly write out this group, using the well-known Wirtinger presentation, which was introduced by Wirtinger in his lectures in Vienna around 1904[28]. Torus knot group and satellite knot group can be written directly. An $(a, b)$-torus knot has group $< u, v \mid u^a = v^b >$. A satellite knot group can be written as an amalgamated free product of the pattern knot group in a torus and the companion knot group, with a free group generated by meridian and longitude amalgamated. Details of those groups can be found in this book[23].

The Wirtinger presentation is built on a *regular projection* of a knot, but it is clear that two presentations of the same knot group under different projections are isomorphic since the knot is completely determined by its complement, and they differ by a finite series of Tietze transformations. For definitions about van Kampen theorem and Tietze transformation, see Lyndon and Schupp[27].

One easy observation we can make about the Wirtinger presentation of a regular projection is that the generators of this presentation correspond to each arc of the diagram. See Figure 4.2 for an example of a $5_2$ knot.

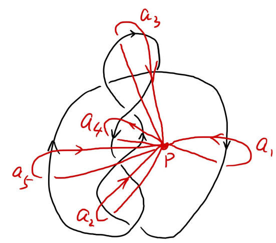


Figure 4.2: Generators of Wirtinger presentation

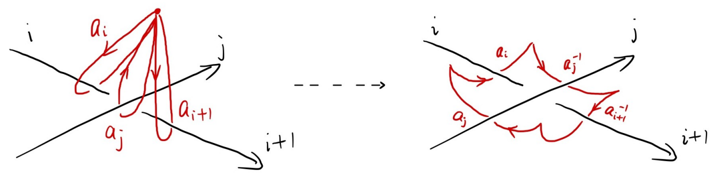


Figure 4.3: Triviality of a relator

**Construction 4.1.** *The* ***Wirtinger presentation*** *of a knot projection is constructed in the following steps:*

1. *Given a regular projection, choose a base-point "above" the projection, write down generators corresponding to arcs.* $a_1, a_2, \ldots, a_n$.

2. *The retraction process of the loop space is given in Figure 4.3, where the direction of each loop follows the right-hand screw rule.*

3. *Write down a relator at each crossing. See Figure 4.4, if it is a crossing of type* $\alpha$*, then write relator* $a_i a_j^{-1} a_{i+1}^{-1} a_j$*, if type* $\beta$*, then write relator* $a_i a_j a_{i+1}^{-1} a_j^{-1}$.

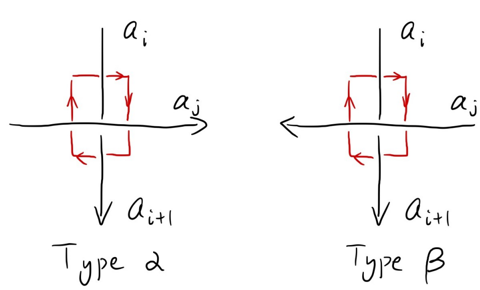


Figure 4.4: Two types of crossings

**Example 4.1.** *The Wirtinger presentation shows that the knot group is not infinite cyclic, hence not trivial. The simplest example for a Wirtinger presentation is that for a* ***trefoil****:*

$$< a_1, a_2, a_3 \mid a_1\, a_3^{-1}\, a_2^{-1}\, a_3, a_3\, a_2^{-1}\, a_1^{-1}\, a_2, a_2\, a_1^{-1}\, a_3^{-1}\, a_1 >$$

*It can also be rewritten as a presentation with two relators:*

$$< a, b \mid a^2 = b^3 >$$

Indeed, since trefoil knot is a torus knot.

## 4.3 The knot complement as a 3-manifold

In the last section, we've seen some topological aspect of the knot complement. However, it can also be viewed as a 3-manifold, especially those knot complements that admit a hyperbolic structure. We will introduce some basic aspects about a knot complement as a 3-manifold, especially in the hyperbolic case.

**Definition 4.12.** *A* ***hyperbolic knot*** *is a non-trivial knot whose complement admits a hyperbolic structure.*

Here a hyperbolic structure means that the knot complement is a manifold with a hyperbolic structure, that is a Riemannian 3-manifold of constant sectional curvature -1, or has some hyperbolic $(X, G)$-structure, where $X = \mathbb{H}^3$, $G = Isom(\mathbb{H}^3)$.

An important theorem by Thurston[11] gives a classification of all knots:

**Theorem 4.3.** *A non-trivial knot is either satellite, torus or hyperbolic.*

The original description of the theorem is: if a knot is neither a torus knot nor a satellite knot, then its complement admits a hyperbolic structure.

**Theorem 4.4.** *The knot complement is a compact 3-manifold.*

A manifold is said to be compact if it is a manifold with boundary, and compact as a topological space. So with this definition, proof for Theorem

4.2 is easy. Note that closed manifold is a manifold without boundary that is closed as topological space, as opposed to a compact manifold. What's even more special about a knot complement is that it has a torus boundary, that is, the boundary of the knot complement is a homeomorphic image of a torus. We will see in the next chapter that with some pre-geometrization 3-manifold topology, we can show some nice(not necessarily useful) properties of the knot complement("Haken-ness").

# CHAPTER 5

# THEOREMS, CONSTRUCTIONS AND CONCLUSION

The topology of 3-manifolds has been studied thoroughly in the last few decades. Unfortunately we only have the capacity for introducing a small part of it. We will roughly use the proof of geometrization conjecture as a line, and try to show what some certain theorems like virtually compact special theorem tell about the knot complement as a compact 3-manifold. In the third section, we will mainly focus on hyperbolic knots. In the last section, we will show some examples of the presentation cube complex of a knot group and its link, and see what Reidemeister moves happen to the complex.

In the first two sections in the chapter, when we talk about compact, orientable, irreducible 3-manifolds, we can keep in mind the knot complement as the most practical example.

## 5.1 3-manifolds before geometrization

In this section, we will develop the essential notions for understanding *the geometrization theorem*, which is a deep decomposition theorem conjectured by Thurston[11], and proven by Perelman[29] using Ricci flow and surgery. Basically, the geometrization says that every compact, irreducible 3-manifold can be cut along incompressible tori, such that the resulting components of it are some basic types of 3-manifolds. There are many ways to express this theorem, we will choose a fashion similar to the JSJ-decomposition theorem, which can be done by replacing hyperbolic by atoroidal. The JSJ-decomposition theorem was introduced and conjectured by Waldhausen[30] and proven by Jaco, Shalen[31] and Johannson[32] independently. Note that the baby form of the *virtual Haken conjecture* was also first conjectured by

Waldhausen[33].
After the decomposition theorems, we will also introduce some properties of 3-manifold group, which were mostly developed before the geometrization was solved.

We will start with the simplest kinds of 3-manifolds, and see how other 3-manifolds can be decomposed into those manifolds.

First, we define what a retract of a group in another group is, similarly as a retract of a topological space:

**Definition 5.1.** *A group $G$ is a* ***retract*** *of a group $H$ if there exists morphisms $\alpha : G \to H$ and $\beta : H \to G$, such that $\alpha \circ \beta = id_G$.*

Here the morphism $\alpha$ is injective, so G is a subgroup of H.

**Lemma 5.1.** *Let $M$ be a compact 3-manifold(with boundary), then $\pi_1(M)$ is a retract of some $\pi_1(N)$, where $N$ is a closed 3-manifold.*

*Proof. sketch*: construct $N$ as the double of $M$ as follows:

$$N = M \cup_{\partial M = \partial M} M$$

This can be viewed as two homeomorphic copies of $M$ glued together along boundaries. □

This lemma tells us that given a compact 3-manifold, we can construct a closed 3-manifold. Therefore, on some level, the study of compact 3-manifolds can be classified into the study of closed 3-manifolds. We will see in the next section many interesting result about closed (hyperbolic) 3-manifolds. However, before that, we still have many implications to make about compact 3-manifolds.

As we have seen in chapter 3, an orientable, prime 3-manifold is almost an irreducible manifold with the exception of $S^1 \times S^2$. For any compact 3-manifold, we have the following decomposition theorem[8]:

**Theorem 5.1.** *Given a compact, orientable, connected 3-manifold $M$, it can be decomposed into a series of connected sum of prime 3-manifolds:*

$$M = P_1 \# P_2 \# \dots \# P_n$$

*The decomposition is unique up to cancellation and insertion of $S^3$'s.*

Another way of decomposing a compact 3-manifold is through a cutting process along some embedded tori, which is the essence of JSJ-decomposition and geometrization. Before actually getting into the decomposition, we need to introduce another important type of basic 3-manifold, so that we can know what our pieces look like.

**Definition 5.2.** *Let $S$ be a surface and $I$ the unit interval. Let $\tau : S \to S$ be a homeomorphism. A 3-manifold $M$ is called an* ***$S^1$-fibered 3-manifold with S fibers*** *if it can be written as $M = S \times I/h$, which is obtained from $S \times I$ by gluing together $(x, 0)$ and $(\tau(x), 1)$, for every $x \in S$. A* ***Seifert fibered 3-manifold*** *$M$ is an $S^1$-fibered 3-manifold with torus fibers.*

The manifold $M$ is called $S^1$-fibered because there is a locally trivial fibration $f : M \to S^1$ with fiber $S$. One can also describe a Seifert fibered 3-manifold $M$ as a compact 3-manifold together with a decomposition of $M$ into disjoint simple closed curves such that each curve has a tubular neighborhood that forms a standard fibered torus[7]. In terms of group action, a Seifert fibered 3-manifold can also be described as a compact orientable 3-manifold $M$ together with an effective action of $S^1$ on it such that no point of $M$ is fixed for all transformations of $S^1$.

**Corollary 5.1.** *A Seifert fibered 3-manifold is not hyperbolic.*

Without surprise, torus knot can be fibered by circles[24], so there's no hyperbolic structure on a torus knot:

**Proposition 5.1.** *A torus knot complement is a Seifert fibered 3-manifold.*

**Definition 5.3.** *A closed 3-manifold is* ***spherical*** *if it admits a complete Riemannian metric of constant curvature +1.*

The universal cover of a spherical 3-manifold is isometric to a 3-sphere. Clearly a knot complement is not spherical.

With all the basic types of manifolds in hand, we can start to decompose the 3-manifolds.

**Theorem 5.2** (Geometrization). *Let $M$ be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. There exists a collection of disjoint, embedded incompressible tori $T_1, \ldots, T_m$ in $M$ such that each component of $M$ cut along $T_1 \cup \ldots \cup T_m$ is either hyperbolic or Seifert fibered. Such collection with a minimal number of tori is unique up to isotopy.*

For a torus knot, its complement is a Seifert fibered manifold. For a hyperbolic knot, there is also no need to do the decomposition. If we replace the word "hyperbolic" by "atoroidal", then the theorem becomes the JSJ-decomposition theorem.

Thurston did a thorough study of 3-dimensional geometry in the 20th century. In his 1982 paper[11], Thurston has conjectured that the interior of every compact 3-manifold has a canonical decomposition into pieces which have geometric structure.

**Definition 5.4.** *A **3-dimensional geometry** is a smooth, simply-connected 3-manifold $X$ together with a smooth, transitive action of a Lie group $G$ by diffeomorphisms on $X$. A **geometric structure** on a 3-manifold $M$ (modeled by $X$) is a diffeomorphism from $M$ to $X/\pi$, where $\pi$ is a discrete subgroup of $G$ that acts freely on $X$. $M$ is said to admit an $(X, G)$**-structure**, or simply is an $X$**-manifold**.*

In this context, the Lie group $G$ is sometimes called *the group of isometries* of $X$, which in the hyperbolic case, is indeed the group of isometries.

Later on, Thurston revolutionized the geometry of 3-manifolds and rewrote the classification of it. It says that if we rule out some simple cases, we can cut a compact, orientable, irreducible 3-manifold $M$ into geometric pieces along disjointly embedded incompressible surfaces that are tori or Klein bottles. Here geometric means the 3-manifold admits one of eight geometries as a geometric structure, see this book for further discussion (Table 1 for a complete classification). One of the eight geometries is hyperbolic geometry, which means hyperbolic 3-manifolds are 3-manifolds with a hyperbolic structure, which coincides with our discussion in chapter 2. The new cutting process, or decomposition theorem, is called the geometric decomposition theorem.

## 5.2 3-manifolds after geometrization

Our goal of this chapter is the virtual Haken conjecture for all closed aspherical 3-manifolds. A well-known important step is the virtually compact special theorem.

To derive and make use of the virtually compact special theorem, the definition of special cube complex needs to be rewritten, which is done in the following proposition by Haglund and Wise[15]:

**Proposition 5.2.** *An NPC cube complex $X$ is special iff there is a graph and a local isometry $X \to S_\Sigma$.*

This local isometry induces an injection on the level of fundamental. Since a covering space of a special cube complex is again a special cube complex, with Proposition 3.2, this proposition shows a classification of subgroups of RAAGs.

**Definition 5.5.** *A finitely generated group is* ***special*** *if it is the fundamental group of an NPC special cube complex.*

Combining this definition with the last proposition, we can see that a special group must be a subgroup of a RAAG. Conversely, a subgroup of a RAAG must be the fundamental group of an NPC special cube complex. All the above discussions can be found in Haglund and Wise[15].

We've defined convexity for a metric space in chapter 2, but this definition can be extended to quasi-convexity:

**Definition 5.6.** *Let $X$ be a geodesic metric space, a subspace $Y$ of $X$ is* ***quasi-convex*** *if there exists a $k \geq 0$, such that any geodesic (segment) in $X$ with endpoints in $Y$ stands in the $k$-neighborhood of $Y$.*

There is also a dual definition of quasi-convexity for subgroup of finitely generated groups, since we have the Cayley graph of a group:

**Definition 5.7.** *Let $G$ be a finitely generated group, with generating set $S$(fixed). A subgroup $H \leq G$ is* ***quasi-convex****(with respect to $S$) if it is a quasi-convex subspace of $Cay_S(G)$.*

With the definition of quasi-convexity, we can show which subgroups are compact special in the next theorem[7]:

**Proposition 5.3.** *A group is compact special iff it is a quasi-convex subgroup of a RAAG.*

Before we get to the virtually compact special theorem for closed hyperbolic 3-manifolds, we can show some easy results for the fundamental group:

**Theorem 5.3.** *Given a closed hyperbolic 3-manifold $N$, the fundamental group $\pi_1(N)$ is word-hyperbolic*

*Proof.* Since hyperbolic 3-space $\mathbb{H}^3$ is Gromov-hyperbolic, the fundamental group of a closed hyperbolic 3-manifold is word-hyperbolic. □

**Definition 5.8.** *An immersed surface $f : F \to M$ in a hyperbolic 3-manifold $M$ is **quasi-Fuchsian** if $\partial\tilde{f}(\tilde{F}) \subset \partial\mathbb{H}^3$ is a topological circle.*

With this definition, we can state the theorem by Kahn and Markovic[34]:

**Theorem 5.4** (Kahn-Markovic)**.** *Every closed hyperbolic 3-manifold contains a dense set of quasi-Fuchsian surface groups.*

According to Bergeron and Wise's work[35], built on Sageev's early results[13], we can further show the relationship between compact NPC cube complex and closed hyperbolic 3-manifold:

**Theorem 5.5.** *The fundamental group of a closed hyperbolic 3-manifold is also the fundamental group of a compact NPC cube complex.*

The next theorem, which generalizes the last theorem, was conjectured by Wise[36], and proven by Agol in his paper solving the virtual Haken conjecture[14]:

**Theorem 5.6.** *Let $G$ be a hyperbolic group which acts properly and cocompactly on a $CAT(0)$ cube complex $X$, then $G$ has a finite index subgroup $H$ such that $X/H$ is special.*

Another way of putting the statement is that $G$ has a finite index subgroup $H$ acting faithfully and specially on $X$.

Theorem 5.6 combined with the geometrization theorem, the work of Kahn and Markovic, one can show that the virtual Haken conjecture is true[14]:

**Theorem 5.7** (virtual Haken conjecture/theorem). *Let $M$ be a closed aspherical 3-manifold. Then there exists a finite-sheeted cover $\tilde{M} \to M$ such that $\tilde{M}$ is Haken.*

In the language of group theory, there is a finite index subgroup of $\pi_1(M)$, which is the fundamental group of a Haken manifold.

There's also a virtually compact special theorem for non-closed(like compact) hyperbolic 3-manifolds in the following theorem by Wise[36]:

**Theorem 5.8.** *The fundamental group of a non-closed hyperbolic 3-manifold is virtually compact special.*

In the case of a hyperbolic knot complement, its knot group is virtually compact special. Although it is compact, it is not special, especially in terms of its presentation complex, which we will show in the section 5.4.

## 5.3 Knots and 3-manifolds

In this chapter, we will discuss what 3-manifold topology can show for the knot complement as a 3-manifold.

Firstly, we discuss this theorem that classifies all knots by their complement. By Thurston's study of hyperbolization of 3-manifolds[11], we have the following theorem:

**Theorem 5.9.** *A knot is either a torus knot, a satellite knot or a hyperbolic knot.*

There is no doubt that hyperbolic knot is the largest class of knot. Readers can also refer to Purcell[24] for a more detailed, complete and recent discussion of this deep theorem.

Another theorem for all knots is given when the asphericity of 3-manifolds is studied:

**Theorem 5.10.** *Nontrivial knot complements are aspherical.*

One proof is given in Burde's book[23] using algebraic topology, where he also points out that the knot complement is a $K(\pi, 1)$-space, which means

its universal cover is contractible.
This theorem can also be shown using the fact that the knot complement is an irreducible 3-manifold, and the knot group is not finite, so by a lemma(1.1.5) in Waldhausen[33], it is aspherical. The irreducibility of a knot complement is shown in the next theorem, together with some other nice properties:

**Theorem 5.11.** *The complement of a non-trivial knot is a compact 3-manifold with toroidal boundary, furthermore, it is Haken.*

*Proof. sketch*
It is clear that the knot complement is a 3-manifold. Compactness follows from the fact that the knot complement is compact as a topological space, and it has a torus boundary.
By definition, Haken means irreducible, compact, and sufficiently large. It is clear that the knot complement cannot be a nontrivial connected sum of two 3-manifolds, so it is prime, plus it is not homeomorphic to $S^1 \times S^2$, it is irreducible. A tubular neighborhood of the knot "further into" the interior of the knot complement is a properly embedded incompressible surface, so the knot complement is sufficiently large. Therefore, the knot complement is Haken. □

However, knot complements with the same fundamental group can be different. In fact, the knot complement is determined by its group together with its peripheral system.

**Theorem 5.12.** *The knot complement is completely determined by the knot group and its peripheral system.*

## 5.4 Cubulating knot groups: experiments and observations

In this chapter, we will discuss some possible properties for knot complements by drawing presentation complexes for knot groups, which, by a theorem in geometric group theory, must be homeomorphic to the knot complement[2]. This means, the knot complement should be topologically "the same" as the presentation complex, which means, all topological information is encoded in the presentation complex, and probably its link. We will see later how

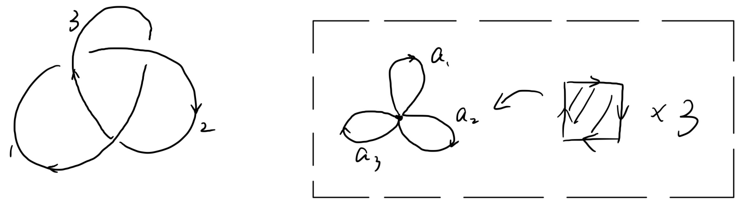


Figure 5.1: Presentation complex of a trefoil knot

Reidemeister moves and other operations change the presentation complex and the link.

By Wirtinger presentation, the knot group is cubical because all relators are four letter words. Therefore, the knot complex is a cube complex, which means we can study the . Moreover, one relator can be cancelled according to the following proposition from combinatorial group theory[6]:

**Proposition 5.4.** *Each relator in the Wirtinger presentation is a consequence(word) of other relators through conjugates.*

We shall start with the simplest example-the trefoil knot.

**Example 5.1.** *As in Figure 5.1, we orient the trefoil first, and then get the presentation of a trefoil:*

$$< a_1, a_2, a_3 \mid a_1\, a_3^{-1}\, a_2^{-1}\, a_3, a_3\, a_2^{-1}\, a_1^{-1}\, a_2, a_2\, a_1^{-1}\, a_3^{-1}\, a_1 >$$

*All three relators are of the same kind. The presentation complex is also in Figure 5.1, where three squares are attached along the words. By definition of the link as an $\epsilon$-sphere in chapter 3, we can draw the link of this presentation complex, as a graph. But it is not convenient to draw, so we should develop some technique to solve this difficulty.*

**Construction 5.1.** *Figure 5.2 shows what the link look like on each square. To draw the link, we first divide the vertex in the presentation complex into two, and then draw the inverses of the edges/generators in the complex, the result is also shown in Figure 5.2. With this picture in hand, we will be able to draw the link by connecting the lines. Finally, by gluing the inverse edges back together with the edges, we get the actual link of the complex(On the*

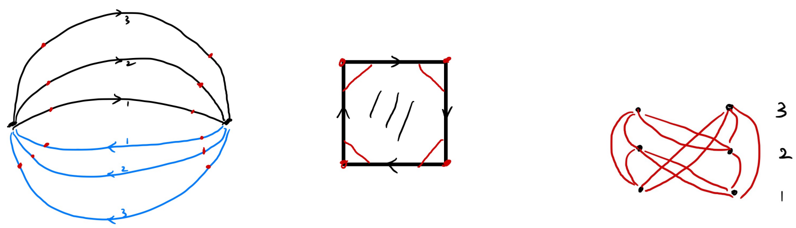

Figure 5.2: Processes in drawing the link of the presentation complex

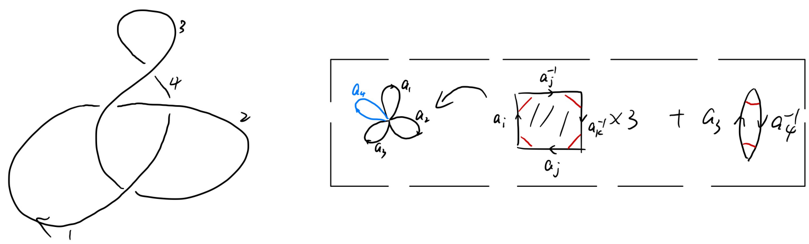

Figure 5.3: Presentation complex after Reidemeister 1

*link, we just need to glue the "opposite" vertices together). The last picture in Figure 5.2 shows how the last process is done.*

In Figure 5.2, $i$ denotes the generator $a_i$, together with the vertices on it. The blue edges denote the inverses, while red edges denote the link.

Applying the same process as above, we can study the change of presentation of the same knot under different Reidemeister moves. Figure 5.3 shows the change of presentation complex after doing the first Reidemeister move to the ordinary projection of the trefoil. It changes by adding one relator together with a bigon. The change of the link compared to the former one is shown in Figure 5.4. The green edges show how the edges are reconnected to the new vertices.

Based on the projection in Figure 5.3, we can further do a third Reidemeister move on it, as is shown in Figure 5.5. The resulting link and the gluing process is shown in Figure 5.6. The comparison of three links we have is shown in Figure 5.7.

The second Reidemeister move can also be done on the original projection of the trefoil. Figure 5.9 shows the change of link after the Reidemeister move. We can see what this move does to the link is just by adding a few

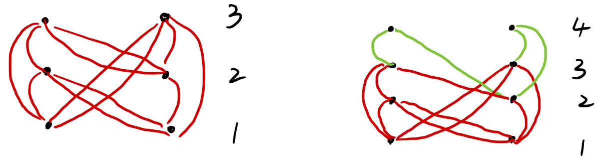


Figure 5.4: Change of link after Reidemeister 1

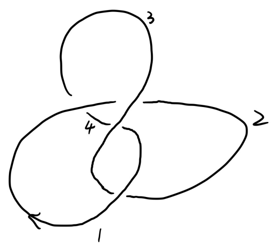


Figure 5.5: Trefoil after first and third Reidemeister moves

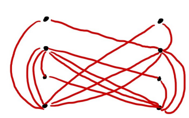

Figure 5.6: link of trefoil after first and third Reidemeister moves

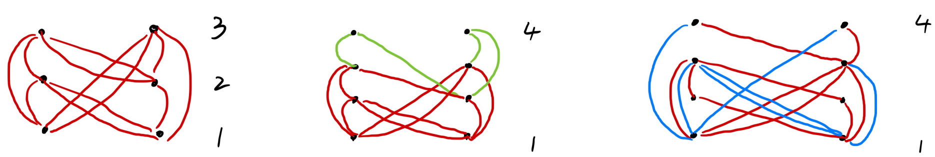


Figure 5.7: The links of trefoil

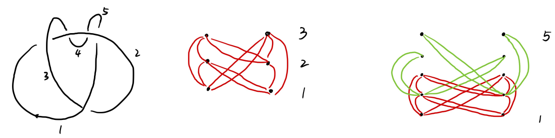


Figure 5.8: Change of link after second Reidemeister move

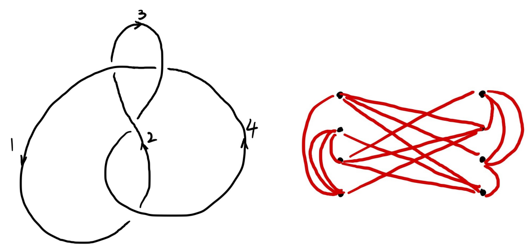


Figure 5.9: The link of a figure 8 knot

redundant edges and vertices to the original link.

Trefoil is the simplest knot, and a torus knot. The next few figures show the links of a hyperbolic knot and a satellite knot. More specifically, Figure 5.9 shows the link of a figure 8 knot, or $4_1$. Figure 5.10 shows the link of a square knot, which is a connected sum of two trefoil knots. It also shows the relation between the link and a trefoil link. Although we have the virtually compact special theorem for non-closed hyperbolic 3-manifolds, the link of figure 8 knot still contains triangles. Therefore, we still need to find its covering space to get a special cube complex.

We notice that in the previous examples, triangle appears once in a while. Therefore, we have the following proposition to show the relation between the appearance of triangles in the link and the "topological triangles" in the regular knot projection(here triangle means a region with three edges and three vertices):

**Proposition 5.5.** *Given a knot projection, if there is no triangle in the projection, or all triangles in the projection are contractible, then there is no triangle with vertices on "both sides" in the link of the knot complex.*

For further work, one can probably study the finite-sheeted covering space of a knot complex using the link. Since we have Theorem 5.8, and hyperbolic knot complement is a non-closed hyperbolic 3-manifold, we might probably be able to find a compact special cover of a hyperbolic knot complex.

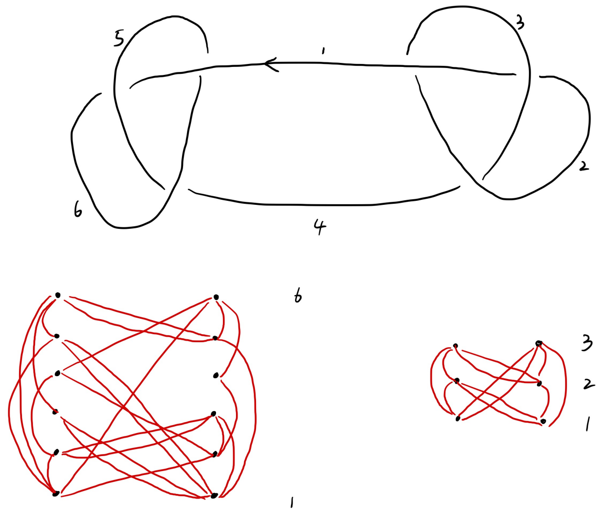

Figure 5.10: The link of a square knot compared to trefoil

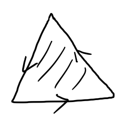

Figure 5.11: A contractible triangle in a knot projection

On the other hand, we can also try to construct some other cube complex that satisfies the NPC link condition. Starting from the Dehn complex in Chapter 3 could be a head start.

# REFERENCES


[1] J. G. Ratcliffe, S. Axler, and K. Ribet, *Foundations of hyperbolic manifolds.* Springer, 1994, vol. 149.

[2] C. Druţu and M. Kapovich, *Geometric group theory.* American Mathematical Soc., 2018, vol. 63.

[3] S. Lang, *Differential and Riemannian manifolds.* Springer Science & Business Media, 2012, vol. 160.

[4] I. Mineyev, "Metric conformal structures and hyperbolic dimension," *Conformal Geometry and Dynamics of the American Mathematical Society*, vol. 11, no. 11, pp. 137–163, 2007.

[5] I. Mineyev, "Flows and joins of metric spaces," *Geometry & Topology*, vol. 9, no. 1, pp. 403–482, 2005.

[6] D. J. Collins, R. I. Grigorchuk, P. F. Kurchanov, and H. Zieschang, *Combinatorial group theory and applications to geometry.* Springer Science & Business Media, 1998, vol. 58.

[7] M. Aschenbrenner, S. Friedl, H. Wilton, and S. Friedl, *3-manifold groups.* European Mathematical Society Zürich, 2015, vol. 20.

[8] A. Hatcher, "Notes on basic 3-manifold topology," 2007.

[9] J. Schultens, *Introduction to 3-manifolds.* American Mathematical Soc., 2014, vol. 151.

[10] W. Haken, "Ein verfahren zur aufspaltung einer 3-mannigfaltigkeit in irreduzible 3-mannigfaltigkeiten," *Mathematische Zeitschrift*, vol. 76, no. 1, pp. 427–467, 1961.

[11] W. P. Thurston, "Three dimensional manifolds, kleinian groups and hyperbolic geometry," *Bulletin of the American Mathematical Society*, vol. 6, no. 3, pp. 357–381, 1982.

[12] W. Haken, "Theorie der normalflächen," *Acta Mathematica*, vol. 105, no. 3-4, pp. 245–375, 1961.

[13] M. Sageev, "Ends of group pairs and non-positively curved cube complexes," *Proceedings of the London Mathematical Society*, vol. 3, no. 3, pp. 585–617, 1995.

[14] I. Agol, D. Groves, and J. Manning, "The virtual haken conjecture," *Doc. Math*, vol. 18, no. 1, pp. 1045–1087, 2013.

[15] F. Haglund and D. T. Wise, "Special cube complexes," *Geometric and Functional Analysis*, vol. 17, no. 5, pp. 1551–1620, 2008.

[16] D. T. Wise, *From Riches to Raags: 3-Manifolds, Right-Angled Artin Groups, and Cubical Geometry: 3-manifolds, Right-angled Artin Groups, and Cubical Geometry.* American Mathematical Soc., 2012, vol. 117.

[17] M. Gromov, "Hyperbolic groups," in *Essays in group theory.* Springer, 1987, pp. 75–263.

[18] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature.* Springer Science & Business Media, 2013, vol. 319.

[19] D. Farley, "A proof of sageev's theorem on hyperplanes in cat (0) cubical complexes," in *Topology and Geometric Group Theory.* Springer, 2016, pp. 127–142.

[20] D. T. Wise, "Subgroup separability of the figure 8 knot group," *Topology*, vol. 45, no. 3, pp. 421–463, 2006.

[21] M. Salvetti, "Topology of the complement of real hyperplanes in cn," *Invent. math*, vol. 88, no. 3, pp. 603–618, 1987.

[22] R. Charney, "An introduction to right-angled artin groups," *Geometriae Dedicata*, vol. 125, no. 1, pp. 141–158, 2007.

[23] G. Burde, H. Zieschang, and M. Heusener, *Knots.* Walter de gruyter, 2013, vol. 5.

[24] J. S. Purcell, *Hyperbolic knot theory.* American Mathematical Soc., 2020, vol. 209.

[25] H. Tietze, "Über die topologischen invarianten mehrdimensionaler mannigfaltigkeiten," *Monatshefte für Mathematik und Physik*, vol. 19, no. 1, pp. 1–118, 1908.

[26] C. McA and J. Luecke, "Knots are determined by their complements," *Journal of the American Mathematical Society*, pp. 371–415, 1989.

[27] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory.* Springer, 2015.

[28] J. Stillwell, *Classical topology and combinatorial group theory.* Springer Science & Business Media, 2012, vol. 72.

[29] G. Perelman, "Ricci flow with surgery on three-manifolds," *arXiv preprint math/0303109*, 2003.

[30] F. Waldhausen, "On the determination of some 3-manifolds by their fundamental group alone," in *Proceedings of the International Symposium on Topology and its Applications: Herceg-Novi, 25.-31. 8. 1968, Yugoslavia*, 1969.

[31] W. Jaco and P. B. Shalen, "Seifert fibered spaces in 3-manifolds," in *Geometric topology.* Elsevier, 1979, pp. 91–99.

[32] K. Johannson, "Équivalences d'homotopie des variétés de dimension 3," *CR Acad. Sci. Paris Sér. AB*, vol. 281, no. 23, pp. A1009–A1010, 1975.

[33] F. Waldhausen, "On irreducible 3-manifolds which are sufficiently large," *Annals of Mathematics*, pp. 56–88, 1968.

[34] J. Kahn and V. Markovic, "Immersing almost geodesic surfaces in a closed hyperbolic three manifold," *Annals of Mathematics*, pp. 1127–1190, 2012.

[35] N. Bergeron and D. T. Wise, "A boundary criterion for cubulation," *American Journal of Mathematics*, vol. 134, no. 3, pp. 843–859, 2012.

[36] D. T. Wise, "Research announcement: the structure of groups with a quasiconvex hierarchy." *Electronic Research Announcements in Mathematical Sciences*, vol. 16, p. 44, 2009.